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On the Finite Simple Groups All of Whose 2-Local Subgroups Are Solvable

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1. INTRODUCTION

A finite group G is said to be of characteristic 2 type if each 2-local subgroup L of G satisfies the condition $C_L(O_2(L)) \subseteq O_2(L)$.

The purpose of this paper is to give an alternative proof of the following theorem, which was a result of the combined work of Janko [1], Smith [2], and Gorenstein and Lyons [3].

MAIN THEOREM. *Let G be a nonabelian simple group of characteristic 2 type, all of whose 2-local subgroups are solvable. Then G is isomorphic to one of the following groups*: $L_2(q)$, $Sz(q)$, $U_3(q)$ (*where q is a power of* 2), $L_2(p)$ (*where p is a Fermat or Mersenne prime with $p \geq 5$*), A_6, $L_3(3)$, M_{11}, $U_3(3)$, ${}^2F_4(2)'$.

Our approach to the proof of the Main Theorem is called an "amalgam method." Several results have already been obtained for our purpose, and we use some of them in the whole of this paper. Therefore we will clarify here what we will assume and what we should prove.

Let X be a group, and let Y be a subgroup of X. By definition, X is Y-irreducible if X has a unique maximal subgroup containing Y. A finite group G is often said to be 2-irreducible if G is S-irreducible for a Sylow

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2-subgroup S of G. Furthermore, a finite group G of even order is said to be thin if $m_{2,p}(G) \leq 1$ for all odd primes p, where the 2-local p-rank $m_{2,p}(G)$ of G is the maximum of the p-ranks $m_p(L)$ as L ranges over the 2-local subgroups of G.

The task of this paper is stated as the following theorem.

THEOREM 1. *Let G be a finite group, all of whose 2-local subgroups are solvable. Let $S \in \mathrm{Syl}_2(G)$, and let H and K be subgroups of G satisfying the following conditions*:

(a) *S is a common Sylow 2-subgroup of H and K.*

(b) *Both H and K are S-irreducible and solvable.*

(c) *No nonidentity subgroup of S is normal both in H and in K.*

(d) $C_H(O_2(H)) \subseteq O_2(H)$ *and* $C_K(O_2(K)) \subseteq O_2(K)$.

Then both H and K are thin.

The next two theorems are assumed as a starting point of our analysis of 2-local subgroups.

THEOREM 2 (Theorems A, C of [4]). *Let G be a finite group of characteristic 2 type, all of whose 2-local subgroups are solvable. Let $S \in \mathrm{Syl}_2(G)$, and assume $O_2(G) = 1$. Then one of the following holds*:

(1) *G has a strongly embedded subgroup.*

(2) *S is dihedral, semidihedral, or isomorphic to a Sylow 2-subgroup of the group $\mathrm{Aut}(Sp_4(2))$.*

(3) *There exist subgroups H and K of G satisfying the above conditions* (a)–(d).

THEOREM 3 (Main Theorem of [5, 6]). *Let (H, S, K) be an amalgam satisfying the above conditions* (a)–(d) *under some embeddings of S into H and K. If both H and K are thin, then the amalgam (H, S, K) or (K, S, H) is isomorphic to a $GL_3(2)$-amalgam, an $Sp_4(2)$-amalgam, a $G_2(2)'$-amalgam, a $G_2(2)$-amalgam, an M_{12}-amalgam, an $\mathrm{Aut}(M_{12})$-amalgam, a ${}^2F_4(2)'$-amalgam, or a ${}^2F_4(2)$-amalgam.*

Let G be a nonabelian simple group satisfying the assumption of the Main Theorem. Then G satisfies the assumption of Theorem 2. If (1) or (2) occurs in Theorem 2, we can appeal to some of the existing classification theorems to identify G with one of the known simple groups. Hence we can assume that (3) occurs in Theorem 2, and then G satisfies the assumption of Theorem 1. Once Theorem 1 is proved, we can know the structures of the subgroups S, H, K by Theorem 3, and then identify G with one of the known simple groups, appealing again to some classification theorems.

Our approach to Theorem 1 pursues and extends the approach to Theorem B of [4], which applies to an alternative proof of the classification of quasi-thin finite simple groups, all of whose 2-local subgroups are solvable. In the course of the proof of Theorem 1, the following two theorems are of particular importance.

THEOREM 4 (Theorem C of [7]). *Let (H, S, K) be an amalgam satisfying the above conditions* (a)–(d), *and the following additional condition* (e):

(e) $S = [O_2(H), O^2(H)](O_2(H) \cap O_2(K))[O_2(K), O^2(K)]$.

Then both H and K are thin.

THEOREM 5 (Section 4.8 of [4]). *Let G be a finite group, all of whose 2-local subgroups are solvable. Let H, S, K, H', S', K' be subgroups of G so that both (H, S, K) and (H', S', K') satisfy the above conditions* (a), (b), (d). *Suppose that $S' = [O_2(H), O^2(H)](O_2(H) \cap O_2(K))[O_2(K), O^2(K)]$, that $H = \langle S, H' \rangle$, and that $K = \langle S, K' \rangle$.*

If (H', S', K') satisfies condition (c), *and both H' and K' are thin, then $(H, S, K) = (H', S', K')$.*

Following [4], we will write $(H', S', K') \preccurlyeq (H, S, K)$ if the assumptions of Theorem 5 are fulfilled. Let G, S, H, K be groups satisfying the assumption of Theorem 1. Then we can actually choose a sequence $(H_0, S_0, K_0), (H_1, S_1, K_1), \ldots, (H_n, S_n, K_n), \ldots$ of amalgams satisfying conditions (a), (b), (d) such that $(H_0, S_0, K_0) = (H, S, K)$ and $(H_{n+1}, S_{n+1}, K_{n+1}) \preccurlyeq (H_n, S_n, K_n)$ for all nonnegative integers n. Since S is finite, we have $S_m = S_{m+1}$ for some nonnegative integer m. If no nonidentity subgroup of S_m is normal in both H_m and K_m, then repeated use of Theorem 5 forces that both H and K are thin. Thus we will assume in our analysis that some nonidentity subgroups of S_m is normal both in H_m and in K_m, in particular that $\langle H_m, K_m \rangle$ is a solvable subgroup of G. In fact, most of the arguments of this paper are devoted to the study of the solvable groups $\langle Z_1, \ldots, Z_n \rangle$ of G generated by D-irreducible subgroups $Z_1, \ldots, Z_n$ with $D \subseteq S$ and $D \in \mathrm{Syl}_2(Z_k)$ for $k = 1, \ldots, n$. Looking at our proof, we can understand that it was sufficient for us to classify the amalgams satisfying not only conditions (a)–(d) but also the additional condition (e).

Related to quadratic 2-subgroups of 2-local subgroups, we will define new characteristic subgroups $N^\infty(n; B)$ for each finite 2-group B and each nonnegative integer n. We will frequently use them throughout the proof of Theorem 1 in the following manner: for a 2-local subgroup L containing a Sylow 2-subgroup S, either $N^\infty(n; S) \trianglelefteq L$ or the structure of $L/O_2(L)$ is highly restricted. Precise definitions and necessary properties are given in Section 2. Section 3 contains necessary information about $GF(2)$-representation of solvable groups, and Sections 4 and 5 are devoted to the body of the proof of Theorem 1.

2. ON FINITE 2-GROUPS

Throughout this section, we will assume that B is an arbitrary finite 2-group, and that n is a nonnegative integer.

DEFINITION 2.1. Let X and Y be subgroups of B.

(1) We denote by $\omega_n(X, Y)$ the set of Y-invariant subgroups V of X possessing the following property: if $[V, L, L] = 1$ for an abelian subgroup L of Y, then there exists a family of generators $\{v_\lambda | \lambda \in \Lambda\}$ of V (depending on L) such that $|L: C_L(v_\lambda)| \leq 2^n$ for all $\lambda \in \Lambda$.

(2) We denote by $\omega_n^*(X, Y)$ the set of elements V of $\omega_{n+1}(X, Y)$ possessing the following property: there exist a family of subgroups $\{V_i | i \in I\}$ of V and a family of subgroups $\{Y_i | i \in I\}$ of Y such that

(a) $V = \langle V_i | i \in I \rangle$,

(b) $V_i \in \omega_n(X, Y_i)$ for all $i \in I$,

(c) $\langle \omega_n(Y, V) \rangle \subseteq Y_i$ for all $i \in I$.

(3) We use the following convention: if Z is a subgroup of B and $\mathscr{B}$ is a set of subgroups of B, then $Z \cap \mathscr{B}$ denotes the set of elements of $\mathscr{B}$ contained in Z.

LEMMA 2.2. *Let $X, X', Y, Y' \subseteq B$ with $X \subseteq X'$ and $Y \subseteq Y'$.*

(1) $X \cap \omega_n(X', Y) \subseteq \omega_n(X, Y) \subseteq \omega_n(X', Y)$ *and* $X \cap \omega_n^*(X', Y) \subseteq \omega_n^*(X, Y) \subseteq \omega_n^*(X', Y)$.

(2) $\omega_n(X, Y) \supseteq \omega_n(X, Y')$ *and* $\omega_n^*(X, Y) \supseteq \omega_n^*(X, Y')$.

(3) $Z(C_X(Y)) \in \omega_n(X, Y) \subseteq \omega_n^*(X, Y) \subseteq \omega_{n+1}(X, Y)$.

Proof. This follows immediately from Definition 2.1. ∎

DEFINITION 2.3. Define a sequence of subgroups $\{N_k(n; B)\}_{k=-1}^{\infty}$ of B recursively as follows:

(1) $N_{-1}(n; B) = B$.

(2) $N_k(n; B) = \langle \omega_n^*(B, N_{k-1}(n; B)) \rangle (k \geq 0)$.

Now, define $N^\infty(n; B) = \bigcap_{k:\ odd} N_k(n; B)$ and $N_\infty(n; B) = \langle N_k(n; B) | k: \mathit{even} \rangle$.

For brevity, we put $N^\infty = N^\infty(n; B)$ and $N_\infty = N_\infty(n; B)$ throughout this section.

LEMMA 2.4. *The following holds*:

(1) $Z(B) \subseteq N_\infty \subseteq N^\infty$. *In particular*, $N^\infty \neq 1$ *whenever* $B \neq 1$.

(2) N^∞ *is a characteristic subgroup of* B.

(3) $\langle \omega_n^*(B, N^\infty)\rangle = N_\infty$ *and* $\langle \omega_n^*(B, N_\infty)\rangle = N^\infty$.

(4) *If* $N^\infty \subseteq Q \subseteq B$, *then* $N^\infty(n; Q) = N^\infty$, *and* N^∞ *is a characteristic subgroup of* Q.

Proof. First, $N_k(n; B)$ is a characteristic subgroup of B for all k because $\mathrm{Aut}(B)$ leaves the set $\omega_n^*(B, N_{k-1}(n; B))$ invariant by induction on k. So (2) holds.

We will see the inclusion relations between the subgroups $N_k(n; B)$. Let $N_k = N_k(n; B)$ for all k. Since $N_{-1} = B \supseteq N_0$, we have $N_{-1} = B \supseteq N_1 = \langle \omega_n^*(B, N_0)\rangle \supseteq \langle \omega_n^*(B, N_{-1})\rangle = N_0$ by Lemma 2.2 (2). Assume that either $N_{i-1} \supseteq N_{i+1} \supseteq N_i$ or $N_{i-1} \subseteq N_{i+1} \subseteq N_i$ holds for some i. Then either $N_i \subseteq N_{i+2} \subseteq N_{i+1}$ or $N_i \supseteq N_{i+2} \supseteq N_{i+1}$ holds by Lemma 2.2 (2) because $N_{k+1} = \langle \omega_n^*(B, N_k)\rangle$ for all k. Thus we can conclude, by induction, that $B = N_{-1} \supseteq N_1 \supseteq \cdots \supseteq N_{2i-1} \supseteq \cdots \supseteq N_{2i} \supseteq \cdots \supseteq N_2 \supseteq N_0$. Since $|B|$ is finite, there exists a nonnegative integer m such that $N_\infty = N_{2i}$ and $N^\infty = N_{2i-1}$ for all $i \geq m$. Hence, together with Lemma 2.2 (3), $Z(B) \subseteq N_0 \subseteq N_\infty = N_{2m} \subseteq N_{2m-1} = N^\infty$, $\langle \omega_n^*(B, N^\infty)\rangle = \langle \omega_n^*(B, N_{2m-1})\rangle = N_{2m} = N_\infty$, $\langle \omega_n^*(B, N_\infty)\rangle = \langle \omega_n^*(B, N_{2m})\rangle = N_{2m+1} = N^\infty$. So (1) and (3) hold.

Suppose that $N^\infty \subseteq Q \subseteq B$. Put $M_k = N_k(n; Q)$. Then $N_{-1} \supseteq M_{-1} \supseteq N^\infty$ by our assumption. Assume that $N_{2i-1} \supseteq M_{2i-1} \supseteq N^\infty$. Then $N_{2i} = \langle \omega_n^*(B, N_{2i-1})\rangle \subseteq \langle \omega_n^*(B, M_{2i-1})\rangle \subseteq \langle \omega_n^*(B, N^\infty)\rangle = N_\infty \subseteq N^\infty \subseteq Q$ by Lemma 2.2 (2) and (1)(3). Thus $\langle \omega_n^*(B, M_{2i-1})\rangle = \langle \omega_n^*(Q, M_{2i-1})\rangle = M_{2i}$ by Lemma 2.2 (1), so $N_{2i} \subseteq M_{2i} \subseteq N_\infty$. Again by Lemma 2.2 (2) and (1)(3), $N_{2i+1} = \langle \omega_n^*(B, N_{2i})\rangle \supseteq \langle \omega_n^*(B, M_{2i})\rangle \supseteq \langle \omega_n^*(Q, M_{2i})\rangle = M_{2i+1}$ and $\langle \omega_n^*(B, N_\infty)\rangle = N^\infty \supseteq Q$, so $M_{2i+1} \supseteq \langle \omega_n^*(Q, N_\infty)\rangle = \langle \omega_n^*(B, N_\infty)\rangle = N^\infty$ by Lemma 2.2 (1). Thus we can conclude, by induction, that $N_{2i-1} \supseteq M_{2i-1} \supseteq N^\infty$ for all i, so $N^\infty(n; Q) = N^\infty$. This proves (4). ∎

LEMMA 2.5. *Let* $V \in \omega_n^*(B, B)$, *and let* $Q \subseteq B$. *If* $|V: C_V(x)| \geq 2^{n+2}$ *for all* $x \in B - Q$, *then* $N^\infty \subseteq Q$.

Proof. Since $V \subseteq \langle \omega_n^*(B, B)\rangle \subseteq \langle \omega_n^*(B, N^\infty)\rangle = N_\infty$ by Lemma 2.2 (2) and Lemma 2.4 (3), we have $N^\infty = \langle \omega_n^*(B, N_\infty)\rangle \subseteq \langle \omega_n^*(B, V)\rangle \subseteq \langle \omega_{n+1}(B, V)\rangle$ by Lemma 2.4 (3) and Lemma 2.2 (2)(3).

Let $L \in \omega_{n+1}(B, V)$. Since $[L, V, V] \subseteq [V, V] = 1$, Definition 2.1 (1) shows that L is generated by the elements x with $|V: C_V(x)| \leq 2^{n+1}$, so $L \subseteq Q$ by our assumption. Therefore $N^\infty \subseteq Q$. ∎

LEMMA 2.6. *Let* $V \in \omega_n(B, B)$, *and let* $Q \subseteq B$. *If* $|V: C_V(x)| \geq 2^{n+1}$ *for all* $x \in B - Q$, *then* $N^\infty \subseteq Q$.

Proof. We have $N^\infty \subseteq \langle \omega_n^*(B,V)\rangle$ by the same argument as in the first paragraph of the proof of Lemma 2.5 because $V \in \omega_n^*(B,B)$ by Lemma 2.2 (3).

Let $L \in \omega_n^*(B,V)$. Let $\{L_i | i \in I\}$ and $\{V_i | i \in I\}$ be families of subgroups of L and V, respectively, satisfying the conditions of Definition 2.1 (2). Let $i \in I$. Since $V \subseteq \langle \omega_n(V,B)\rangle \subseteq \langle \omega_n(V,L)\rangle$ by Lemma 2.2 (1)(2), we have $V \subseteq V_i$ by Definition 2.1 (2c), so $L_i \in \omega_n(B,V)$ by Definition 2.1 (2b). Since $[L_i, V, V] \subseteq [V,V] = 1$, Definition 2.1 (1) shows that $L_i = \langle x \in L_i | |V: C_V(x)| \leq 2^n \rangle \subseteq Q$, so $L \subseteq Q$ by Definition 2.1 (2a). Therefore $N^\infty \subseteq Q$. ▮

3. ON $GF(2)$-REPRESENTATION OF SOLVABLE GROUPS

In this section, we study $GF(2)$-representation of solvable groups.

First, we review the basic structure of 2-irreducible solvable groups.

LEMMA 3.1. *Let G be a 2-irreducible solvable group with $S \in \mathrm{Syl}_2(G)$, and let M be the unique maximal subgroup of G containing S. Put $Q = O_2(G)$. Then the following holds*:

(1) *If $N \trianglelefteq G \neq SN$ (or, equivalently, $O^2(G) \not\subseteq N \trianglelefteq G$), then $S \cap N \trianglelefteq G$, $Q \in \mathrm{Syl}_2(QN)$, and G/N is 2-irreducible and solvable.*

(2) *$G = O_{2,p,2}(G)$ for some odd prime p.*

Let $P \in \mathrm{Syl}_p(G)$.

(3) *S/Q acts irreducibly and faithfully on $PQ/\Phi(P)Q$.*

(4) *$Z(S/Q)$ is cyclic, and an element $t \in S$ inverts $PQ/\Phi(P)Q$ if and only if $Q \neq tQ \in \Omega_1(Z(S/Q))$.*

(5) *$P \cap M = \Phi(P)$.*

(6) *Let $x \in S - Q$. Then $x \notin M$ if and only if $G = \langle S, x\rangle$.*

(7) *$O^2(G) = P[Q, O^2(G)]$ and $[Q, O^2(G)] = S \cap O^2(G)$.*

Proof. Parts (1)–(6) are proved in (2.1) of [11]. Since $[Q,P] \subseteq [Q, O^2(G)]$, we have $P[Q,O^2(G)] \trianglelefteq PQ = O_{2,p}(G)$, so $O^2(G) = P[Q,O^2(G)]$. Thus $[Q,O^2(G)] = Q \cap O^2(G) = S \cap O^2(G)$, so (7) holds. ▮

Throughout the remainder of this section, we will assume that G is a finite solvable group and that V is a nontrivial irreducible faithful $GF(2)G$-module. Let $S \in \mathrm{Syl}_2(G)$, and put $S[k] = \langle x \in S | |V: C_V(x)| \leq 2^k \rangle$ for all positive integers k.

LEMMA 3.2. $S[1] \subseteq O_{2',2}(G)$.

Proof. We may assume that $S[1] \neq 1$. Then $S[1] \subseteq \langle \mathscr{P}(G,V)\rangle$, where $\mathscr{P}(G,V)$ is the set of nonidentity elementary abelian subgroups A of G satisfying $|A|\,|C_V(A)| \geq |B|\,|C_V(B)|$ for all subgroups B of A. We can see $\langle \mathscr{P}(G,V)\rangle \subseteq O_{2',2}(G)$ by (3.2) and (3.3) of [8] because G is solvable and $O_2(G) \subseteq C_G(V) = 1$. ∎

LEMMA 3.3. *Suppose that G is 2-irreducible and that $S[2] \neq 1$. Let $\mathscr{H}$ be the set of the homogeneous components of V_P, where $P = O(G)$. Then there exists a normal subgroup P_W for each $W \in \mathscr{H}$ such that $P = \langle P_W | W \in \mathscr{H}\rangle, W = [V, P_W]$, and one of the following holds*:

(1) $P_W \cong Z_3$, $|W| = 2^2$, *and* $S[1] \neq 1$.

(2) $P_W \cong Z_3$, $|W| = 2^2$, *and* $S[1] = 1$.

(3) $P_W \cong Z_5$, $|W| = 2^4$, *and* $S[1] = 1$.

(4) $P_W \cong Esp_{27}$, $|W| = 2^6$, *and* $S[1] = 1$.

Proof. We first note that $P \in \mathrm{Syl}_p(G)$ for some odd prime p by Lemma 3.1 (2) because $O_2(G) = 1$. Let $t \in S$ be an involution with $|V{:}\ C_V(t)| \leq 2^2$, and put $\mathscr{T} = t^S$. Let $u \in \mathscr{T}$, and define $P_u = [P, u]$ and $V_u = [V, P_u]$. Then $P_u \trianglelefteq P$ for all $u \in \mathscr{T}$. Since $|\Omega_1(Z(S))| = 2$ by Lemma 3.1 (4) and $\langle \mathscr{T}\rangle \trianglelefteq S$, we have $\Omega_1(Z(S)) \subseteq \langle \mathscr{T}\rangle$, and then $P = \langle P_u | u \in \mathscr{T}\rangle$, because $\Omega_1(Z(S))$ inverts $P/\Phi(P)$ also by Lemma 3.1 (4). Hence $V = \sum_{u \in \mathscr{T}} V_u$. Let $\mathscr{I}$ be a subset of $\mathscr{T}$ that is maximal subject to the following condition: if $u \neq u'$, then $V_u \neq V_{u'}$ for all $u, u' \in \mathscr{I}$. Then $P = \langle P_u | u \in \mathscr{I}\rangle$, and $V = \sum_{u \in \mathscr{I}} V_u$. Let $u \in \mathscr{I}$. Pick an irreducible submodule X of V_P contained in V_u. Then $X = [X, P_u]$ as $P_u \trianglelefteq P$. So, if $W \in \mathscr{H}$ contains X, then $W = [W, P_u] \subseteq [V, P_u] = V_u$. This shows that V_u is a direct sum of some members of $\mathscr{H}$ as V_P is completely reducible.

Suppose that V_u is homogeneous itself. Then the set $\{V_u | u \in \mathscr{I}\}$ coincides with the set $\mathscr{H}$. For $W \in \mathscr{H}$, choose the unique element $u \in \mathscr{I}$ so that $W = V_u$, and define $P_W = P_u$. Then $P = \langle P_W | W \in \mathscr{H}\rangle$ and $W = [V, P_W]$ for each $W \in \mathscr{H}$. The structures of W and P_W are described in Lemma 2.8 of [11] because $|W{:}\ C_W(u)| \leq |V{:}\ C_V(u)| \leq 2^2$. In particular, $Z(P_W)$ is cyclic as W is homogeneous and $C_{P_W}(W) \subseteq C_P(V) = 1$. Since G is irreducible on V, $N_G(W)$ is also irreducible on W, so the case $|W| = 2^4$ and $P_W \cong Z_3$ is eliminated. Hence one of the three cases (1), (3), (4) occurs.

Suppose that V_u is not homogeneous. For $W \in \mathscr{H}$, choose $u \in \mathscr{I}$ so that $W \subseteq V_u$. Note that u does not centralize any member of $\mathscr{H}$ contained in V_u as $C_{V_u}(P_u) = 0$. Thus there exists $W' \in \mathscr{H}$ such that $V_u = W + W'$, where $|W| = |W'| = 2^2$, because $|V_u{:}\ C_{V_u}(u)| \leq |V{:}\ C_V(u)| \leq 2^2$. Hence one of the two cases (1), (2) occurs, and, moreover, $P/C_P(W) \cong P/C_P(W') \cong Z_3$, $C_P(V_u) = C_P(W) \cap C_P(W')$, and u inverts $P/C_P(V_u)$. Since V_u is not homogeneous, Lemma 2.8 of [11] shows that $P_u \cong E_9$, and so $P = P_u C_P(V_u)$. Define $P_W = P_u \cap C_P(W')$. Then $P \trianglerighteq P_W \cong Z_3$ and $[V, P_W] = W$. Since $C_P(V) = 1$ and $|W| = 2^2$, we can conclude that P_W is a unique subgroup of P with the property $W = [V, P_W]$, and so, in particular, P_W is independent on the choice of $u \in \mathscr{I}$. Thus $P_u = P_W P_{W'}$, and hence $P = \langle P_W | W \in \mathscr{H} \rangle$. ∎

LEMMA 3.4. *Let $L \subseteq S$ with $[V, L, L] = 0$. Then $V = \langle v \in V | |L: C_L(v)| \leq 2 \rangle$.*

Proof. Since $O_2(G) = 1$, there exists an odd prime p such that $[O_p(G), L] \neq 1$. Put $P = O_p(G)$. Let W be a homogeneous component of V_P. Put $N = N_G(W)$, $C = C_G(W)$, and $\overline{N} = N/C$. If $L \not\subseteq N$, then $|L: C \cap L| \leq 2$ by (3.2) of [7]. Suppose that $L \subseteq N$. If $W = V$, then $|L: C \cap L| \leq 2$ by (3.3) of [7] because $[V, L, L] = 0 \neq [V, [N, L]]$ for a $GF(2)PL$-module V. So suppose that $W \subset V$. Then $W = \langle w \in W | |L: C_L(w)| \leq 2 \rangle$ by the inductive hypothesis, because $\overline{N}$ is irreducible on W and $[W, \overline{L}, \overline{L}] \subseteq [V, L, L] = 0$. Hence the lemma holds because V is a direct sum of the homogeneous components of V_P. ∎

4. PRELIMINARIES

Throughout the remainder of this paper, let G be a finite group satisfying the assumption of Theorem 1, and let S be a Sylow 2-subgroup of G.

We begin with some definitions.

DEFINITION 4.1. Let $D \subseteq S$.

(1) We denote by $\Delta(D)$ the set of subgroups Z of G satisfying the following conditions:

(a) $D \in \mathrm{Syl}_2(Z)$.

(b) Z is D-irreducible and solvable.

(c) $C_Z(O_2(Z)) \subseteq O_2(Z)$.

(2) Let $D \subseteq S$ and $X, Y \in \Delta(D)$. We write $X \subseteq_2 Y$ and say that X is 2-embedded into Y if $[O_2(X), O^2(X)] \subseteq O_2(Y)$.

(3) We denote by Γ the set of triplets (X, D, Y) with the conditions $D \subseteq S$ and $X, Y \in \Delta(D)$. Let $\beta = (X, D, Y) \in \Gamma$. Define $O_2(\beta) = O_2(\langle X, Y \rangle) \cap D$. Note that $O_2(\beta)$ is the largest subgroup of D that is normal in both X and Y. We call β thin if both X and Y are thin. Let $\beta' = (X', D', Y') \in \Gamma$. We write $\beta' \preccurlyeq \beta$ if $D' = [O_2(X), O^2(X)](O_2(X) \cap O_2(Y))[O_2(Y), O^2(Y)]$, $X = \langle X', D \rangle$, and $Y = \langle Y', D \rangle$.

(4) Let L be a subgroup of G with $R = L \cap S \in \mathrm{Syl}_2(L)$. Let $D \subseteq R$. We denote by $\Gamma_L(D)$ the set of subgroups Z of L satisfying the following conditions:

(a) $D \in \mathrm{Syl}_2(Z)$.

(b) Z is D-irreducible.

(c) $\langle O^2(Z), R\rangle = L$.

LEMMA 4.2. *Let $D \subseteq R \subseteq S$ and $L \in \Delta(R)$ with $[O_2(L), O^2(L)] \subseteq D$.*

(1) $\Gamma_L(D) \neq \emptyset$.

Let $Z \in \Gamma_L(D)$.

(2) $[O_2(Z), O^2(Z)] = [O_2(L), O^2(Z)]$.

(3) $C_Z(O_2(Z)) \subseteq C_Z([O_2(Z), O^2(Z)]) \subseteq O_2(Z)$.

(4) $Z \in \Delta(D)$.

(5) $\bigcap O_2(Z)^R \subseteq O_2(L) \supseteq \bigcap_{X \in \Gamma_L(D)} O_2(X)$.

(6) $\langle O^2(Z)^R\rangle = O^2(L) = \langle O^2(X) | X \in \Gamma_L(D)\rangle$.

Proof. We first note that $D \in \mathrm{Syl}_2(DO^2(L))$ as $O^2(L) \cap R = [O_2(L), O^2(L)] \subseteq D$ by Definition 4.1 (1a) and Lemma 3.1 (7), and that $DO^2(L)$ is generated by the set Γ' of D-irreducible subgroups X of L with $D \in \mathrm{Syl}_2(X)$.

Let M be the unique maximal subgroup of L containing R. Let $X \in \Gamma'$. Then $X \in \Gamma_L(D)$ if and only if $X \not\subseteq M$ by Lemma 3.1 (6). Put $L^* = L/[O_2(L), O^2(L)]$. Then $O^2(L)^*$ has odd order by Lemma 3.1 (7), so $O^2(L)^* \cap M^* = \Phi(O^2(L)^*)$ by Lemma 3.1 (5). Hence we have $\Gamma_L(D) = \{X \in \Gamma' | O^2(X)^* \not\subseteq \Phi(O^2(L)^*)\}$. Thus $O^2(L) = \langle O^2(X) | X \in \Gamma'\rangle = \langle O^2(X) | X \in \Gamma_L(D)\rangle$, so (1) holds. Moreover, we have $L = \langle O^2(Z)^R\rangle R$ by Definition 4.1 (4c), so (6) holds.

Let $Q = [O_2(Z), O^2(Z)]$ and $Y = O^2(C_Z(Q))$. Part (2) holds because $Q = O_2(Z) \cap D \subseteq O_2(L) \cap R = [O_2(L), O^2(L)]$ by Lemma 3.1 (7). Since $[O_2(L), Y] \subseteq [O_2(L), O^2(Z)] = Q$ by (2), we have $Y \subseteq C_L(O_2(L)) \subseteq O_2(L)$ by Definition 4.1 (1c). Hence $Y = 1$, so (3) holds. Part (4) is a consequence of (3) and Definition 4.1 (4a)(4b).

Let $\bar{L} = L/O_2(L)$, and $\Gamma'' = Z^R$ or $\Gamma_L(D)$. Then $\overline{O^2(L)} = O(\bar{L})$ by Definition 4.1 (1b) and Lemma 3.1 (2). Since $[\overline{O_2(X)}, \overline{O^2(X)}] \subseteq \overline{O_2(X)} \cap O(\bar{L}) = 1$ for all $X \in \Gamma''$, we have $[\bigcap_{X \in \Gamma''} \overline{O_2(X)}, O(\bar{L})] = [\bigcap_{X \in \Gamma''} \overline{O_2(X)}, \langle \overline{O^2(X)} | X \in \Gamma''\rangle] = 1$ by (6). Thus $\bigcap_{X \in \Gamma''} \overline{O_2(X)} \subseteq C_{\bar{R}}(O(\bar{L})) = 1$, so (5) holds. ∎

LEMMA 4.3. *Let $D \subseteq R \subseteq S$ and $L, L' \in \Delta(R)$ with $[O_2(L), O^2(L)][O_2(L'), O^2(L')] \subseteq D$ and $L \not\subseteq_2 L'$.*

(1) *If* $Y \in \Gamma_L(D)$, *then there exists* $X \in \Gamma_{L'}(D)$ *such that* $Y \not\subseteq_2 X$.

(2) *If* $Y \in \Gamma_{L'}(D)$, *then there exists* $Z \in \Gamma_L(D)$ *such that* $Z \not\subseteq_2 Y$.

Proof. We first note that $\emptyset \neq \Gamma_L(D) \subseteq \Delta(D) \supseteq \Gamma_{L'}(D) \neq \emptyset$ by Lemma 4.2 (1)(4).

Let $Y \in \Gamma_L(D)$. Suppose that $Y \subseteq_2 X$ for all $X \in \Gamma_{L'}(D)$. Then $[O_2(L), O^2(Y)] = [O_2(Y), O^2(Y)] \subseteq O_2(X)$ for all $X \in \Gamma_{L'}(D)$ by Lemma 4.2 (2). Thus $[O_2(L), O^2(Y)] \subseteq \bigcap_{X \in \Gamma_{L'}(D)} O_2(X) \subseteq O_2(L')$ by Lemma 4.2 (5). Hence $[O_2(L), O^2(L)] \subseteq [O_2(L), \langle O^2(Y)^R \rangle] \subseteq O_2(L')$ by Lemma 4.2 (6), a contradiction. So (1) holds.

Let $Y \in \Gamma_{L'}(D)$. Suppose that $Z \subseteq_2 Y$ for all $Z \in \Gamma_L(D)$. Then $[O_2(L), O^2(Z)] = [O_2(Z), O^2(Z)] \subseteq O_2(Y)$ for all $Z \in \Gamma_L(D)$ by Lemma 4.2 (2). Thus $[O_2(L), O^2(L)] \subseteq [O_2(L), \langle O^2(Z) | Z \in \Gamma_L(D) \rangle] \subseteq O_2(Y)$ by Lemma 4.2 (6). Hence $[O_2(L), O^2(L)] \subseteq \bigcap O_2(Y)^R \subseteq O_2(L')$ by Lemma 4.2 (5), a contradiction. So (2) holds. ∎

LEMMA 4.4. *Let* $E \subseteq D \subseteq R \subseteq S$ *and* $L \in \Delta(R)$. *Suppose* $Z \in \Gamma_L(D)$ *with* $[O_2(Z), O^2(Z)] \subseteq E$. *Then* $[\Omega_1(Z(E)), O^2(Z)] = 1$ *if and only if* $\Omega_1(Z(R)) \subseteq Z(L)$.

Proof. Put $Q = O_2(L)$, $T = [O_2(Z), O^2(Z)]$, $U = \Omega_1(Z(Q))$, $W = \Omega_1(Z(E \cap Q))$. Then $[Q, O^2(Z)] = T = O^2(Z) \cap D \subseteq E \cap Q$ by Lemma 4.2 (2) and Lemma 3.1 (7), so $[U, O^2(Z)] \subseteq [W, O^2(Z)]$, and $|O^2(Z)(E \cap Q) : (E \cap Q)|$ is odd. Hence $C_{[W, O^2(Z)]}(O^2(Z)) = 1$.

Suppose first $[\Omega_1(Z(E)), O^2(Z)] = 1$. Then $[W, O^2(Z)] \cap \Omega_1(Z(E)) = 1$, so $[W, O^2(Z)] = 1$. Hence $[U, O^2(Z)] = 1$, and then $\Omega_1(Z(R)) \subseteq Z(L)$ by Definition 4.1 (4c) because $\Omega_1(Z(R)) \subseteq U$ by Lemma 4.2 (3).

Suppose next $[\Omega_1(Z(E)), O^2(Z)] \neq 1$. Since $\Omega_1(Z(E)) \subseteq C_Z(T) \subseteq O_2(Z)$ by Lemma 4.2 (3), we have $[\Omega_1(Z(E)), O^2(Z)] \subseteq T \subseteq E \cap Q$, so $1 \neq [\Omega_1(Z(E)), O^2(Z)] \subseteq [W, O^2(Z)]$.

Here, let $T \subseteq N \trianglelefteq M \subseteq Q$, and suppose $V_N = [\Omega_1(Z(N)), O^2(Z)] \neq 1$. We will show that $V_M = [\Omega_1(Z(M)), O^2(Z)] \neq 1$. By Lemma 3.1 (2), Z is a $\{2, p\}$-group for some odd prime p. Let $P \in \mathrm{Syl}_p(Z)$. Then $O^2(Z) = PT$ by Lemma 3.1 (7), and $M = NC_M(P)$ because $[M, P] \subseteq [Q, P] \subseteq T \subseteq N$. Thus $V_N = [\Omega_1(Z(N)), P] \trianglelefteq M$ and $C_{V_N}(P) = 1$. Hence $1 \neq [C_{V_N}(M), P] \subseteq [\Omega_1(Z(M)), P] \subseteq V_M$.

Now, since $T \subseteq E \cap Q \trianglelefteq \trianglelefteq Q$ and $[W, O^2(Z)] \neq 1$, we have $[U, O^2(Z)] \neq 1$, so $[U, O^2(L)] \neq 1$. We have $C_{[U, O^2(L)]}(O^2(L)) = 1$ because $|O^2(L)Q : Q|$ is odd by Lemma 3.1 (2). Therefore $\Omega_1(Z(R)) \not\subseteq Z(L)$ because $1 \neq C_{[U, O^2(L)]}(R) \subseteq \Omega_1(Z(R))$. ∎

LEMMA 4.5. *Let* $D \subseteq S$ *and* $X \in \Delta(D)$. *Let* N *be a solvable subgroup of* G *containing* X, *and let* M *be a normal subgroup of* N.

(1) *If* $O^2(X) \not\subseteq M$, *then* $M \cap D \trianglelefteq X$.

(2) *If* M *is maximal subset to the condition that* $O^2(X) \not\subseteq M \trianglelefteq N$, *then* $[O_2(X), O^2(X)] \subseteq M \cap D$.

(3) *Let* $Y \in \Delta(D)$ *with* $Y \subseteq N$. *If* $O^2(X) \not\subseteq M \supseteq O^2(Y)$, *then* $Y \subseteq_2 X$.

Proof. Part (1) follows from Lemma 3.1 (1). For the proof of (2), let Q be a preimage of a minimal normal subgroup of N/M. The maximality of M forces that $O^2(X) \subseteq Q$, so $|Q/M|$ is odd by the solvability of N. Thus $[O_2(X), O^2(X)] = O^2(X) \cap D \subseteq Q \cap D \subseteq M \cap D$ by Lemma 3.1 (7), so (2) holds. Part (3) holds because $[O_2(Y), O^2(Y)] = O^2(Y) \cap D \subseteq M \cap D \subseteq O_2(X)$ by Lemma 3.1 (7) and (1). ∎

LEMMA 4.6. *Let* $D \subseteq S$ *and* $\Delta \subseteq \Delta(D)$. *Suppose that* Δ *is a finite set and that* $\langle \Delta \rangle$ *is solvable. Then there exists an element* $Y \in \Delta$ *such that* $Y \subseteq_2 X$ *for all* $X \in \Delta$.

Proof. We will first show that if $Z_1 \not\subseteq_2 Z_2 \not\subseteq_2 \cdots \not\subseteq_2 Z_k$ for $Z_1, Z_2, \dots, Z_k \in \Delta$, then $Z_k \subseteq_2 Z_i$ for $1 \leq i \leq k$. Put $N = \langle \Delta \rangle$. Take a normal subgroup M of N that is maximal subject to the condition $O^2(Z_k) \not\subseteq M$. Then $O^2(Z_i) \not\subseteq M$ for $1 \leq i \leq k$ by repeated use of Lemma 4.5 (3). So $M \cap D \subseteq O_2(Z_i)$ for $1 \leq i \leq k$ by Lemma 4.5 (1). On the other hand, $[O_2(Z_k), O^2(Z_k)] \subseteq M \cap D$ by Lemma 4.5 (2).

Suppose the lemma is false. If $Z_1 \not\subseteq_2 Z_2 \not\subseteq_2 \cdots \not\subseteq_2 Z_{j-1}$ for $Z_1, Z_2, \dots, Z_{j-1} \in \Delta$, then we can pick $Z_j \in \Delta$ such that $Z_{j-1} \not\subseteq_2 Z_j$ and $Z_j \neq Z_i$ for $1 \leq i \leq j$. Hence, by induction, all elements of Δ are arranged in such a way that $Z_1 \not\subseteq_2 Z_2 \not\subseteq_2 \cdots \not\subseteq_2 Z_n$. But then $Z_n \subseteq_2 Z_i$ for $1 \leq i \leq n$, a contradiction. ∎

LEMMA 4.7. *Let* $\beta = (X, D, Y) \in \Gamma$. *Suppose either that* β *is not thin or that* $O_2(\beta) \neq 1$. *Put* $D' = [O_2(X), O^2(X)](O_2(X) \cap O_2(Y))[O_2(Y), O^2(Y)]$, *and let* $X' \in \Gamma_X(D')$ *and* $Y' \in \Gamma_Y(D')$. *Then either* $X' \subseteq_2 Y'$ *or* $Y' \subseteq_2 X'$.

Proof. Put $D_0 = D$, $X_0 = X$, $Y_0 = Y$, $D_1 = D'$, $X_1 = X'$, $Y_1 = Y'$. Inductively, put

$$D_n = \left[O_2(X_{n-1}), O^2(X_{n-1})\right]\left(O_2(X_{n-1}) \cap O_2(Y_{n-1})\right) \times \left[O_2(Y_{n-1}), O^2(Y_{n-1})\right],$$

and choose $X_n \in \Gamma_{X_{n-1}}(D_n)$ and $Y_n \in \Gamma_{Y_{n-1}}(D_n)$. Note that this is possible by Lemma 4.2 (1). Define $\beta_n = (X_n, D_n, Y_n)$ for all nonnegative integers n. Then we have $\beta_n \in \Gamma$ and $\beta_{n+1} \preccurlyeq \beta_n$ for all $n \geq 0$ by Lemma 4.2 (4) and Definition 4.1 (3)(4c). Since D is finite, we have $D_m = D_{m+1}$ for some nonnegative integer m. Suppose that $O_2(\beta_m) = 1$. Then β_m is thin by

Theorem 4. Hence repeated use of Theorem 5 forces that $\beta_m = \beta_{m-1} = \cdots = \beta_1 = \beta_0$. This contradiction shows that $O_2(\beta_m) \neq 1$. Thus $\langle X_m, Y_m \rangle$ is solvable, and so either $X_m \subseteq_2 Y_m$ or $Y_m \subseteq_2 X_m$ by Lemma 4.6.

Let $2 \leq j \leq m$, and suppose that $X_j \subseteq_2 Y_j$. Then $O_2(Y_j) \supseteq [O_2(X_j), O^2(X_j)] = O^2(X_j) \cap D_j = O^2(X_j) \cap D_{j-2}$ by Lemma 3.1 (7). Since

$$\begin{aligned}\left[O_2(X_{j-2}), O^2(X_j)\right] &\subseteq \left[O_2(X_{j-2}), O^2(X_{j-1})\right] = \left[O_2(X_{j-1}), O^2(X_{j-1})\right]\\ &\subseteq O_2(X_{j-1})\end{aligned}$$

by Lemma 4.2 (2), we have $[O_2(X_{j-2}), O^2(X_j)] \subseteq [O_2(X_{j-1}), O^2(X_j)] = [O_2(X_j), O^2(X_j)] \subseteq O^2(X_j)$, also by Lemma 4.2 (2). Thus

$$\begin{aligned}\left[O_2(X_{j-1}), O^2(X_j)\right] &= \left[O_2(X_j), O^2(X_j)\right]\\ &\subseteq \bigcap O_2(Y_j)^{[O_2(X_{j-2}), O^2(X_{j-2})](O_2(X_{j-2}) \cap O_2(Y_{j-2}))}\\ &\subseteq \bigcap O_2(Y_j)^{D_{j-1}}\left[O_2(Y_{j-2}), O^2(Y_{j-2})\right] \subseteq O_2(Y_{j-1})\end{aligned}$$

by Lemma 4.2 (5). Hence we conclude that

$$\begin{aligned}\left[O_2(X_{j-1}), O^2(X_{j-1})\right] &\subseteq \left[O_2(X_{j-1}), \langle O^2(X_j)^{D_{j-1}}\rangle\right]\\ &= \langle\left[O_2(X_{j-1}), O^2(X_j)\right]^{D_{j-1}}\rangle\\ &= \langle\left[O_2(X_j), O^2(X_j)\right]^{D_{j-1}}\rangle \subseteq O_2(Y_{j-1})\end{aligned}$$

by Lemma 4.2 (6), so $X_{j-1} \subseteq_2 Y_{j-1}$. Similarly, $Y_j \subseteq_2 X_j$ implies $Y_{j-1} \subseteq_2 X_{j-1}$. Therefore, by induction on j, we have either $X_1 \subseteq_2 Y_1$ or $Y_1 \subseteq_2 X_1$. ∎

LEMMA 4.8. *Let $\beta = (X, D, Y) \in \Gamma$. If either $X \subseteq_2 Y$ or $Y \subseteq_2 X$, then $O_2(\beta) \neq 1$.*

Proof. By the symmetry between X and Y, we may assume that $X \subseteq_2 Y$. Put $D' = O_2(Y)$. Then $\Gamma_X(D') \neq \emptyset$ by Lemma 4.2 (1), and D acts on $\Gamma_X(D')$ as $D' \trianglelefteq D$. Let $\{X_1, \dots, X_n\}$ be a D-orbit on $\Gamma_X(D')$. Then $C_{X_k}(O_2(X_k)) \subseteq O_2(X_k)$ for $1 \leq k \leq n$ and $O^2(X) = \langle O^2(X_1), \dots, O^2(X_n)\rangle$ by Lemma 4.2 (3)(6).

Let $1 \leq k \leq n$. We may assume that no nonidentity characteristic subgroup of D' is normal in X_k because $X = \langle X_k, D\rangle$ by Definition 4.1 (4c). Therefore, by the Corollary of [9], X_k is described as follows: $X_k = D'J(X_k)$, $J(X_k) = L_k \times D_k \times E_k$, $L_k \cong S_4 \times \cdots \times S_4$, $D_k \cong D_8 \times \cdots \times D_8$, E_k is an elementary abelian 2-group. Put $V_k = [O_2(X_k), O^2(X_k)]$. Then

$V_k = O^2(X_k) \cap D' \subseteq J(D')$ by Lemma 3.1 (7) and the structure of X_k. Put $V = [O_2(X), O^2(X)]$. Then $V = [O_2(X), \langle O^2(X_k) | 1 \leq k \leq n\rangle] = \langle [O_2(X), O^2(X_k)] | 1 \leq k \leq n\rangle = \langle [O_2(X_k), O^2(X_k)] | 1 \leq k \leq n\rangle = \langle V_k | 1 \leq k \leq n\rangle \subseteq J(D')$ by Lemma 4.2 (2). We may assume that $V \subset J(D')$. Let A be a Hall $2'$-subgroup of $D'O^2(Y)$, and put $W = \langle V^A\rangle$. Then $W \trianglelefteq DA = Y$. If $[V, W] = 1$, then $[O_2(X), O^2(X)] = V \subseteq W \subseteq C_{D'}(V) \subseteq O_2(X)$ by Lemma 3.1 (1), so $W \trianglelefteq \langle O^2(X), Y\rangle = \langle X, Y\rangle$.

Suppose that $[V, W] \neq 1$. Then we have $[V_i, V_j^a] \neq 1$ for some i, j, and $a \in A$. Let $\gamma = (X_i, D', X_j^a)$. Then $\gamma \in \Gamma$ by Lemma 4.2 (4). Moreover, we have $X_i \not\subseteq_2 X_j^a$ because $J(D') \supseteq V_i \not\subseteq C_{J(D')}(V_j^a) = O_2(J(X_j^a)) = J(D') \cap O_2(X_j^a)$ by the structure of X_j. Put $E = J(D')$. Since $V_iV_j^a \subseteq E$, there exist $Z_1 \in \Gamma_{X_i}(E)$ and $Z_2 \in \Gamma_{X_j^a}(E)$ such that $(Z_1, E, Z_2) \in \Gamma$ and $Z_1 \not\subseteq_2 Z_2$ by Lemma 4.2 (1)(4) and Lemma 4.3. Let $\delta = (Z_1, E, Z_2)$. Since $E \ntrianglelefteq X_i = \langle Z_1, D'\rangle$, we have $E \ntrianglelefteq Z_1$, so $Z_1 = \langle E^{Z_1}\rangle$ by Lemma 3.1 (1). Hence $[O_2(Z_1), O^2(Z_1)] \subseteq [O_2(Z_1), \langle E^{Z_1}\rangle] = \langle [O_2(Z_1), E]^{Z_1}\rangle \subseteq \langle [E, E]^{Z_1}\rangle \subseteq \langle [\Omega_1(Z(E))^{Z_1}\rangle \subseteq \Omega_1(Z(O_2(Z_1)))$ by the structure of E and Lemma 4.2 (3), so $C_E([O_2(Z_1), O^2(Z_1)]) = O_2(Z_1)$ by Lemma 3.1 (1). Similarly, $C_E([O_2(Z_2), O^2(Z_2)]) = O_2(Z_2)$. Thus $Z_1 \not\subseteq_2 Z_2$ implies $Z_2 \not\subseteq_2 Z_1$. Next, put $E' = [O_2(Z_1), O^2(Z_1)](O_2(Z_1) \cap O_2(Z_2))[O_2(Z_2), O^2(Z_2)]$. Then there exist $Z_1' \in \Gamma_{Z_1}(E')$ and $Z_2' \in \Gamma_{Z_2}(E')$ such that $(Z_1', E', Z_2') \in \Gamma$ and $Z_1' \not\subseteq_2 Z_2'$ by Lemma 4.2 (1)(4) and Lemma 4.3. Since $Z_2 \not\subseteq_2 Z_1 = \langle Z_1', E\rangle$, we have $E' \ntrianglelefteq Z_1'$. Similarly, $E' \ntrianglelefteq Z_2'$. Thus, as above, we have $C_{E'}([O_2(Z_i'), O^2(Z_i')]) = O_2(Z_i')$ for $i = 1, 2$, and then $Z_1' \not\subseteq_2 Z_2'$ implies $Z_2' \not\subseteq_2 Z_1'$. Now, Lemma 4.7 shows that δ is thin and that $O_2(\delta) = 1$. Since $[O_2(Z_1), O^2(Z_1)] \subseteq \Omega_1(Z(O_2(Z_1)))$, Theorem 3 yields that δ is isomorphic either to a $GL_3(2)$-amalgam or to an $Sp_4(2)$-amalgam, and then $|E| \leq 2^4$. This forces $|V| \leq 2^3$, and so X is thin and $O^2(X)D'$ is 2-irreducible. Now, by Proposition 2.3 of [11], some nonidentity A-invariant subgroup, say W', of D' is normal in $O^2(X)D'$. So $\langle W'^D\rangle$ is normal in both X and Y as $X = O^2(X)D$ and $Y = DA$. ∎

5. THE PROOF OF THEOREM 1

We will begin the proof of Theorem 1. Assume that G and S continue to satisfy the assumption of Theorem 1. Let H and K be subgroups of G satisfying conditions (a)–(d) stated in Theorem 1.

Put $\alpha = (H, S, K)$.

LEMMA 5.1. $\alpha \in \Gamma$, $H \not\subseteq_2 K$, $K \not\subseteq_2 H$.

Proof. This follows from Lemma 4.8 and conditions (a)–(d) of Theorem 1. ∎

By the symmetry between H and K, we will assume the following:

ASSUMPTION 5.2. $\Omega_1(Z(S)) \not\subseteq Z(H)$.

LEMMA 5.3. *The group H has a noncentral minimal normal subgroup.*

Proof. Let $W = \langle \Omega_1(Z(S))^H \rangle$. Then $O_2(H) \subseteq C_H(W) \not\supseteq O^2(H)$ by Assumption 5.2. Thus $[W, O^2(H)] \neq 1$, and $O^2(H)C_H(W)/C_H(W)$ has odd order by Lemma 3.1 (1)(2), so $C_{[W, O^2(H)]}(O^2(H)) = 1$. Therefore a minimal normal subgroup of H contained in $[W, O^2(H)]$ is noncentral. ∎

Let V be a noncentral minimal normal subgroup of H. Put $T^* = [O_2(H), O^2(H)](O_2(H) \cap O_2(K))[O_2(K), O^2(K)]$, $T = O_2(H)[O_2(K), O^2(K)]$, and $D[k] = \langle t \in D \mid |V: C_V(t)| \leq 2^k \rangle$ for all subgroups D of S and all positive integers k.

LEMMA 5.4. *Let $T^* \subseteq D \subseteq S$. Then the following holds for all positive integers k:*

(1) $V \in \omega_k(D, D)$.

(2) $N^\infty(k; D) \subseteq D[k]$.

(3) *If $D \trianglelefteq S$, then $N^\infty(k; D) \not\subseteq O_2(H) \cap O_2(K)$.*

Proof. Let k be a positive integer.

We first note that $V = [V, O^2(H)] \subseteq [O_2(H), O^2(H)] \subseteq T^* \subseteq D$. Let L be an abelian subgroup of D with $[V, L, L] = 1$. Put $\overline{H} = H/C_H(V)$, and regard V as a nontrivial irreducible faithful $GF(2)\overline{H}$-module. Since $[V, \overline{L}, \overline{L}] = [V, L, L] = 1$, Lemma 3.4 shows that $V = \langle v \in V \mid |\overline{L}: C_{\overline{L}}(v)| \leq 2 \rangle = \langle v \in V \mid |L: C_L(v)| \leq 2 \rangle$. Hence $V \in \omega_1(D, D)$, and then $V \in \omega_k(D, D)$ by Lemma 2.2 (3). Put $Q = D[k]$. Then $|V: C_V(x)| \geq 2^{k+1}$ for all $x \in D - Q$ by the definition of Q, so we have $N^\infty(k; D) \subseteq Q$ by Lemma 2.6. So (1) and (2) hold.

Suppose that $D \trianglelefteq S$. Since $1 \neq T \subseteq D$, we have $N^\infty(k; D) \neq 1$ by Lemma 2.4 (1). Let $L = H$ or K. Then $D \cap O_2(L) \trianglelefteq SO^2(L) = L$ because $[O_2(L), O^2(L)] \subseteq D \cap O_2(L) \trianglelefteq S$. Therefore, if $N^\infty(k; D) \subseteq O_2(L)$, then $N^\infty(k; D) = N^\infty(k; D \cap O_2(L)) \trianglelefteq L$ by Lemma 2.4 (4)(2). Hence (3) holds. ∎

The next lemma is a technical one that shows special relations between elements of $\Delta(D)$ where $D \subseteq S$.

LEMMA 5.5. *Let $D \subseteq S$ and $n > 1$. Take $Z_1, \dots, Z_n \in \Gamma_H(D) \cup \Gamma_K(D)$, and put $N = \langle Z_1, \dots, Z_n \rangle$. Assume that $Z_n \not\subseteq_2 Z_{n-1} \not\subseteq_2 \cdots \not\subseteq_2 Z_2 \not\subseteq_2 Z_1$,* *that N is solvable, and that $Z_1 \in \Gamma_H(D)$. Then the following hold*:

(1) *There exists an abelian normal subgroup U of N such that $U \subseteq D$ and $|U\colon C_U(x)| \geq 2^2$ for all $x \in D - O_2(Z_n)$.*

(2) *If $D \trianglelefteq S$, $|S\colon D| \leq 2^4$, and $Z_n \in \Gamma_L(D)$ where $L = H$ or K, then $N^\infty(1; D) \subseteq O_2(L)$.*

Proof. Take a subgroup M of N that is maximal subject to the condition $O_2(Z_1) \not\subseteq M \trianglelefteq N$. Then $M \subset N$, and $M \cap D \trianglelefteq \langle Z_1, \dots, Z_n \rangle$ by Lemma 4.5 (1)(3). Put $P = O(N \bmod M)$. Since $O_2(N/M) = 1$ by the maximality of M, we have $P \cap D = M \cap D \subseteq O_2(N) \subseteq M \subset P$, so $O^2(Z_1) \subseteq P$ also by the maximality of M. Let $I = \langle O^2(Z_1)^N \rangle$, $E = I \cap D$, $W = \Omega_1(Z(E))$. Then $[O_2(Z_1), O^2(Z_1)] = O^2(Z_1) \cap D \subseteq E \subseteq D$ by Lemma 3.1 (7). Since $E = I \cap P \cap D = I \cap M \cap D$ and $O^2(Z_1) \not\subseteq I \cap M$, we have $E \trianglelefteq N$ by Lemma 4.5 (1)(3). Hence $[W, O^2(Z_1)] \neq 1$ by Lemma 4.4 because $Z_1 \in \Gamma_H(D)$ and $\Omega_1(Z(S)) \not\subseteq Z(H)$. Thus $1 \neq [W, I] \trianglelefteq N$.

Next, we will show that if $|S\colon D| \leq 2^4$, then $C_{[W, I]}(I) = 1$. Take a subgroup R of N that is maximal subject to the conditions $R \trianglelefteq N$ and $R \cap D \in \mathrm{Syl}_2(R)$. Since $O(N/R) = 1$ by the maximality of R, $C_{N/R}(O_2(N/R)) \subseteq O_2(N/R)$ by the solvability of N/R. Let $Q = O_2(N \bmod R)$. Suppose that $O^2(Z_1) \not\subseteq Q$. Then $Q \cap D \trianglelefteq N$ by Lemma 4.5 (1)(3), so $(Q \cap D)R \trianglelefteq N$ and $Q \cap D \in \mathrm{Syl}_2((Q \cap D)R)$. Hence $Q \cap D \subseteq R$ by the maximality of R. Let W_0 be a chief factor of N within Q/R so that $O^2(Z_1) \not\subseteq C = C_N(W_0)$. Let S' be a subgroup between $(N \cap S)R$ and N so that $S'/R \in \mathrm{Syl}_2(N/R)$. Then $|S'|/|R\colon R \cap D| \leq |S|$, so $|W_0| \leq |Q/R| = |QD\colon RD| \leq |S'|/|RD| \leq |S|/|D| \leq 2^4$. Hence N/C is isomorphic to a subgroup of $GL_4(2)$. Also we have $O_2(N/C) = 1$ because N is irreducible on W_0. Thus $N/C = O_{2',2}(N/C)$ by inspecting the solvable subgroups of $GL_4(2)$. Since $O^2(Z_1) \not\subseteq C$, we have $C \cap D \trianglelefteq Z_1$ by Lemma 4.5 (1). Thus $[O_2(Z_2), O^2(Z_2)] = O^2(Z_2) \cap D \subseteq O(N \bmod C) \cap D = C \cap D \subseteq O_2(Z_1)$ by Lemma 3.1 (7), which shows that $Z_2 \subseteq_2 Z_1$, a contradiction. Thus $O^2(Z_1) \subseteq C_N(Q/R) \subseteq Q$, and then $O^2(Z_1) \subseteq R$. Hence $I \subseteq R$, so $E = O_2(I) \in \mathrm{Syl}_2(I)$. Therefore $C_{[W, I]}(I) = 1$.

Now, define a subgroup U of N as follows: U is a minimal normal subgroup of N contained in $[W, I]$ if $|S\colon D| \leq 2^4$; $U = [W, I]$ otherwise. Let U_0 be a chief factor of N within U so that $U_0 = [U_0, I]$. Put $C' = C_N(U_0)$. Since $O^2(Z_1) \not\subseteq C'$, $C' \cap D \subseteq O_2(N) \cap D = M \cap D$. Since N is irreducible on U_0, $O_2(N/C') = 1$. If $O^2(Z_n) \subseteq O(N \bmod C')$, then $[O_2(Z_n), O^2(Z_n)] = O^2(Z_n) \cap D \subseteq O(N \bmod C') \cap D = C' \cap D \subseteq M \cap D \subseteq O_2(Z_{n-1})$ by Lemma 3.1 (7), which shows that $Z_n \subseteq_2 Z_{n-1}$, a contradiction. Thus $O^2(Z_n) \not\subseteq O(N \bmod C')$, so $O^2(Z_n) \not\subseteq O_{2',2}(N \bmod C')$, and hence $O_{2',2}(N \bmod C') \cap D \subseteq O_2(Z_n)$ by Lemma 4.5 (1). Therefore $|U\colon C_U(x)| \geq |U_0\colon C_{U_0}(x)| \geq 2^2$ for all $x \in D - O_2(Z_n)$ by Lemma 3.2, so (1) holds. Suppose that $|S: D| \le 2^4$. Then Lemma 3.4 implies $U \in \omega_1(D, D)$, as in the proof of Lemma 5.4 (1), so $N^\infty(1; D) \subseteq O_2(Z_n)$ by Lemma 2.6. Hence, if $D \trianglelefteq S$ and $Z_n \in \Gamma_L(D)$ where $L = H$ or K, we have $N^\infty(1; D) \subseteq \bigcap O_2(Z_n)^S \subseteq O_2(L)$ by Lemma 4.2 (5), so (2) holds. ∎

In the remainder of the proof, we will assume that α is not thin and derive a contradiction.

LEMMA 5.6. *Let $X \in \Gamma_H(T)$. Then there exists $Y \in \Gamma_K(T)$ such that $X \subseteq_2 Y \not\subseteq_2 X$ and $\langle X, Y \rangle$ is solvable.*

Proof. Let $X \in \Gamma_H(T)$. Then there exists $Y \in \Gamma_K(T)$ such that $Y \not\subseteq_2 X$ by Lemma 4.3 (2). Put $X^* = O^2(X)T^*$. Then $T^* \in \mathrm{Syl}_2(X^*)$ because $O^2(X) \cap T = [O_2(X), O^2(X)] = [O_2(H), O^2(X)] \subseteq [O_2(H), O^2(H)] \subseteq T^*$ by Lemma 3.1 (7) and Lemma 4.2 (2). Moreover, as $T = O_2(H)T^*$, there is a one-to-one correspondence between the set of subgroups of X containing T and the set of subgroups of X^* containing T^*. Thus X^* is 2-irreducible, and then $X^* \in \Gamma_X(T^*)$. Hence there exists $Y^* \in \Gamma_Y(T^*)$ such that $Y^* \not\subseteq_2 X^*$ as above, because $[O_2(Y), O^2(Y)] = [O_2(K), O^2(Y)] \subseteq [O_2(K), O^2(K)] \subseteq T^*$. Since $X^* \in \Gamma_H(T^*)$ and $Y^* \in \Gamma_K(T^*)$, we have $X^* \subseteq_2 Y^*$ by Lemma 4.7. Thus $[O_2(X), O^2(X)] = O_2(X) \cap T = O_2(X^*) \cap T^* = [O_2(X^*), O^2(X^*)] \subseteq O_2(Y^*)$ by Lemma 3.1 (7), and so $[O_2(X), O^2(X)] \subseteq \bigcap O_2(Y^*)^T \subseteq O_2(Y)$ by Lemma 4.2 (5). Hence we have $X \subseteq_2 Y$. Thus $O_2(\langle X, Y \rangle) \ne 1$ by Lemma 4.8, and then $\langle X, Y \rangle$ is solvable. ∎

LEMMA 5.7. $V \subseteq O_2(K)$.

Proof. By Lemma 5.6, we can choose $X \in \Gamma_H(T)$ and $Y \in \Gamma_K(T)$ so that $Y \not\subseteq_2 X$ and $\langle X, Y \rangle$ is solvable. Then, by Lemma 5.5 (1), there exists an abelian normal subgroup U of $\langle X, Y \rangle$ such that $U \subseteq T$ and $|U: C_U(x)| \ge 2^2$ for all $x \in T - O_2(Y)$. Since $V \in \omega_1(T, T)$ by Lemma 5.4 (1), and $[V, U, U] \subseteq [U, U] = 1$, we have $V = \langle v \in V \,|\, |U: C_U(v)| \le 2 \rangle \subseteq O_2(Y)$, so $V \subseteq \bigcap O_2(Y)^S \subseteq O_2(K)$ by Lemma 4.2 (5). ∎

LEMMA 5.8. $\Omega_1(Z(S)) \subseteq Z(K)$.

Proof. Suppose false. Then, as in Lemma 5.3, K has a noncentral minimal normal subgroup, say, W. Furthermore, $W \subseteq O_2(H)$ by Lemma 5.7 and the symmetry between H and K. Now, we will assume that $Q(K(S)) \ntrianglelefteq H$, where $Q(K(S))$ is a characteristic subgroup of S defined in [10]. Then H has a unique noncentral chief factor within $O_2(H)$ as in the proof of Proposition 2.2 of [11]. Thus we have $[O_2(H), O^2(H)] = V$. Hence $H \subseteq_2 K$ by Lemma 5.7, a contradiction. ∎

Let $Y \in \Gamma_K(T)$. Define $U = \langle V^Y \rangle$, $Z = \bigcap V^Y$, $T^+ = O_2(H)[O_2(Y), O^2(Y)]$, and $Y^+ = O^2(Y)T^+$.

LEMMA 5.9. *If* $T[1] \subseteq O_2(H)$, *then the following hold*:

(1) *U is normal abelian 2-subgroup of Y contained in T^+.*

(2) $[Z, T^+] = 1$.

(3) *Y^+ is 2-irreducible with $T^+ \in \mathrm{Syl}_2(Y^+)$.*

(4) $T^+ = C_{T^+}(V)[O_2(Y^+), O^2(Y^+)]$.

Proof. First, U is a normal 2-subgroup of Y by Lemma 5.7, so $U \subseteq O_2(Y) \subseteq T$. Let $y \in Y$. Then $V^y \in \omega_1(T^y, T^y)$ by Lemma 5.4 (1). Since $[V^y, V, V] \subseteq [V, V] = 1$, we have $V^y = \langle w \in V^y \,|\, |V\colon C_V(w)| \le 2\rangle \subseteq T[1] \subseteq O_2(H) \subseteq T^+$. Thus $U \subseteq T^+$ and U is abelian, so (1) holds.

Since $H \not\leq_2 K = \langle S, Y\rangle$ by Lemma 5.1 and Definition 4.1 (4c), and $T = O_2(H)[O_2(K), O^2(K)] = C_T(V)O_2(Y) = C_T(Z)O_2(Y)$, we have $T \ne O_2(Y)$, and then $Y = \langle T^Y\rangle = \langle C_T(Z)^Y\rangle O_2(Y)$ by Lemma 3.1 (1). Thus $[O_2(Y), O^2(Y)] \subseteq O^2(Y) \subseteq \langle C_T(Z)^Y\rangle$. Hence $T^+ \subseteq C_T(Z)$, so (2) holds.

Since $T = T^+[O_2(K), O^2(K)]$ and $[[O_2(K), O^2(K)], O^2(Y)] = [O_2(Y), O^2(Y)] = T \cap O^2(Y) \subseteq T^+$ by Lemma 4.2 (2) and Lemma 3.1 (7), we conclude that (3) holds and that $[O_2(Y), O^2(Y)] = [O_2(Y^+), O^2(Y^+)]$ as in the proof of Lemma 5.6. So (4) also holds. ∎

LEMMA 5.10. *If* $T[1] \subseteq O_2(H)$, *then* $U \in \omega_2(T^+, T^+)$.

Proof. Suppose first that $[V, T^+] \not\subseteq Z$. Let $V^{(1)} = C_{V \bmod Z}(T^+)$, $V^{(2)} = C_{V \bmod V^{(1)}}(T^+)$, and $U^{(i)} = \langle (V^{(i)})^{Y^+}\rangle$ for $i = 1, 2$. Then $Z \subset V^{(1)} \subset V^{(2)}$ by our assumption, and so $[U^{(2)}, [O_2(Y^+), O^2(Y^+)]] \not\subseteq Z$. By Lemma 5.9 (4). Suppose that $[U, L, L] = 1$ for an abelian subgroup L of T^+. Let $L' = L \cap O_2(Y^+)$. Note that Y^+ is a 2-irreducible solvable group acting on an abelian 2-group $U^{(2)}$ by Lemma 5.9 (3)(1). Thus $|L\colon L'| \le 2$ by (3.9) of [7]. Since $V^y \in T^+ \cap \omega_1(T^y, T^y) \subseteq \omega_1(T^+, O_2(Y^+))$ for all $y \in Y^+$ by Lemma 2.2 (1)(2), we have $U \in \omega_1(T^+, O_2(Y^+))$ by (2.2) of [7], so $U = \langle u \in U \,|\, |L'\colon C_{L'}(u)| \le 2\rangle = \langle u \in U \,|\, |L\colon C_L(u)| \le 2^2\rangle$. Thus $U \in \omega_2(T^+, T^+)$ by Lemma 5.9 (1).

Suppose next that $[V, T^+] \subseteq Z$. Let $\Gamma' = \{X \in \Gamma_H(T^+) \,|\, T^+ \ntrianglelefteq X\}$, and define $V_X = C_V(O_2(X))$ for $X \in \Gamma_H(T^+)$. Then $V = \langle V_X \,|\, X \in \Gamma_H(T^+)\rangle = \langle V_X \,|\, X \in \Gamma'\rangle$ because $\bigcap_{X \in \Gamma_H(T^+)}[V^*, O_2(X)] = [V^*, \bigcap_{X \in \Gamma_H(T^+)} O_2(X)] \subseteq [V^*, O_2(H)] = 1$ for the dual module V^* of V.

Let $X \in \Gamma'$, and put $U_X = \langle V_X^{Y^+}\rangle$. Then it suffices to prove that $U_X \in \omega_2(T^+, T^+)$ by Lemma 5.9 (1) and (2.2) of [7] because $U = \langle U_X \,|\, X \in \Gamma'\rangle$. Since $[V, (T^+)^2] \subseteq [V, T^+, T^+] = 1$, and $C_T(V) = O_2(H)$ by (3.1)(1), $T^+/O_2(H)$ is elementary abelian. Thus $|T^+/O_2(X)| = 2$ by Lemma 3.1 (4) because $O_2(H) \subseteq O_2(X) \subset T^+$. Choose a family of subgroups $\{V_X^{(i)} \,|\, i \in I\}$

of V_X so that the following hold for all $i \in I$

(1) $V_X^{(i)}$ is a T^+-invariant subgroup of order 2 or 2^2.

(2) $V_X^{(i)} \not\subseteq Z$.

(3) $|V_X^{(i)}: Z_X^{(i)}| = 2$ where $Z_X^{(i)} = [V_X^{(i)}, T^+]$.

Let $i \in I$, and put $U_X^{(i)} = \langle (V_X^{(i)})^{Y^+} \rangle$. Then $U_X^{(i)}/Z_X^{(i)} \in \omega_1(T^+/Z_X^{(i)}, T^+/Z_X^{(i)})$ by (3.10) of [7] because $V_X^{(i)}/Z_X^{(i)} \subseteq \Omega_1(Z(T^+/Z_X^{(i)}))$. Thus $U_X^{(i)} \in \omega_2(T^+, T^+)$ by (2.3) of [7] because $|Z_X^{(i)}| \leq 2$. Moreover, we have $Z \in \omega_0(T^+, T^+) \subseteq \omega_2(T^+, T^+)$ by Lemma 5.9 (2) and Lemma 2.2 (3). Therefore $U_X = \langle U_X^{(i)} | i \in I \rangle Z \in \omega_2(T^+, T^+)$ by (2.2) of [7]. ∎

LEMMA 5.11. *The following hold*:

(1) $T[2] \not\subseteq O_2(H)$.

(2) *If* $T[1] \subseteq O_2(H)$, *then* $T = T[2]$.

Proof. To prove (1) and (2), we may assume that $T[1] \subseteq O_2(H)$. Then $N^\infty(1; T) \subseteq T[1] \subseteq O_2(H) \subseteq T^+$ by Lemma 5.4 (2), so $N^\infty(1; T^+) = N^\infty(1; T) \subseteq O_2(H)$ by Lemma 2.4 (4).

Let $\min_{x \in T - O_2(Y)} |U: C_U(x)| = 2^d$, and let $x \in T - O_2(Y)$ with $x^2 \in O_2(Y)$ and $|U: C_U(x)| = 2^d$. Since $T = O_2(H)[O_2(K), O^2(K)] = C_T(V)O_2(Y)$, there exists an element $y \in C_T(V)$ such that $y \equiv x \pmod{O_2(Y)}$. Let P be a Hall $2'$-subgroup of Y. Take an element $g \in P - \Phi(P)$ such that $k = [g, y]^{-1} \notin \Phi(P)O_2(Y)$. Then $2^d = |U: C_U(x)| \geq |V^{g^{-1}}: V^{g^{-1}} \cap V^{g^{-1}x}| = |V^{g^{-1}}: V^{g^{-1}} \cap V^{g^{-1}y}| = |V^{g^{-1}}: V^{g^{-1}} \cap V^{g^{-y}}| = |V: V \cap V^k|$. Since $y \in C_T(V)$, we have $[V \cap V^k, \langle y, y^k \rangle] = 1$ so $[V \cap V^k, O_2(Y), \langle y, y^k \rangle] = 1 = [\langle y, y^k \rangle, V \cap V^k, O_2(Y)]$ as $V, V^k \trianglelefteq O_2(Y)$. Hence $[O_2(Y), \langle y, y^k \rangle, V \cap V^k] = 1$ by the three-subgroup lemma, so $[O_2(Y), \langle y, y^k \rangle] \subseteq T[d]$. Since $\langle y, y^k \rangle \not\subseteq \Phi(P)O_2(Y)$, we have $O^2(Y) = \langle \langle y, y^k \rangle^T \rangle$, so $[O_2(Y), O^2(Y)] \subseteq T[d]$. Therefore $[O_2(K), O^2(K)] = [O_2(K), \langle O^2(Y)^S \rangle] = \langle [O_2(K), O^2(Y)]^S \rangle = \langle [O_2(Y), O^2(Y)]^S \rangle \subseteq \langle T[d]^S \rangle = T[d]$ by Lemma 4.2 (6)(2), and so $T = C_T(V)[O_2(K), O^2(K)] = T[d]$.

Suppose that $d \geq 2$. Let $L \in \omega_1(T^+, U)$. Since $[L, U, U] \subseteq [U, U] = 1$, we have $L = \langle x \in L \,|\, |U: C_U(x)| \leq 2 \rangle$, so $L \subseteq O_2(Y)$. Thus $\langle \omega_1(T^+, U) \rangle \subseteq T^+ \cap O_2(Y) = O_2(Y^+)$. Hence $U \in \omega_1^*(T^+, T^+)$ by Definition 2.1 (2) because $V^y \in T^+ \cap \omega_1(T, O_2(Y)) \subseteq \omega_1(T^+, O_2(Y^+))$ for all $y \in Y^+$, and $U = \langle V^y | y \in Y^+ \rangle \in \omega_2(T^+, T^+)$.

Now, suppose that $d \geq 3$. Then we have $N^\infty(1; T^+) \subseteq O_2(Y^+) \subseteq O_2(Y)$ by Lemma 2.5, so $N^\infty(1; T) \subseteq \bigcap O_2(Y)^S \subseteq O_2(K)$. This contradicts Lemma 5.4 (3), and hence $d \leq 2$. Therefore (1) and (2) hold. ∎

Let $\overline{H} = H/C_H(V)$, and regard V as a nontrivial irreducible faithful $GF(2)\overline{H}$-module. Put $P = O_{2,2'}(H)$. Then we have $\overline{P} = O(\overline{H}) = \overline{O^2(H)}$ by Lemma 3.1 (1)(2).

LEMMA 5.12. *The group T acts intransitively on the set $\mathscr{H}$ of homogenous components of $V_{\overline{P}}$.*

Proof. First, assume that $T[1] \not\subseteq O_2(H)$. Let $X \in \Gamma_H(T)$. Since $T[1] \trianglelefteq S$, and $\bigcap O_2(X)^S \subseteq O_2(H)$ by Lemma 4.2 (5), there exists an element $t \in T - O_2(X)$ such that $|V: C_V(t)| = 2$. Since $O^2(X) \not\subseteq C_X(V)$ by Lemma 4.2 (6), we have $C_X(V) \subseteq \Phi(O_{2,2'}(X) \bmod O_2(X))$ by Lemma 3.1 (1)(2)(5), and so $[O^2(X), t] \not\subseteq C_X(V)$. Thus there exists a unique element $W \in \mathscr{H}$ such that $1 \neq \overline{[O^2(X), t]} \subseteq [\overline{P}, \bar{t}] = \overline{P}_W \cong Z_3$ by Lemma 3.3, so $\overline{X} \supseteq \overline{P}_W$, and $\overline{P}$ is an elementary abelian 3-group by Lemma 3.3. Hence $C_H(V) = \Phi(P \bmod O_2(H))$ as above. Suppose that T is transitive on $\mathscr{H}$. Then $\overline{X} \supseteq \langle \overline{P}_W | W \in \mathscr{H} \rangle = \overline{P} = \overline{O^2(H)}$, so $O^2(H) \subseteq O^2(X)C_H(V) = O^2(X)O_2(H) \subseteq O^2(X)T = X$. Thus $O^2(X) = O^2(H)$. By Lemma 5.6, there exists an element $Y \in \Gamma_T(K)$ such that $X \subseteq_2 Y$. Then $[O_2(H), O^2(H)] \subseteq O^2(H) \cap T = O^2(X) \cap T = [O_2(X), O^2(X)] \subseteq O_2(Y)$, by Lemma 3.1 (7), and hence $[O_2(H), O^2(H)] \subseteq \bigcap O^2(Y)^S \subseteq O_2(K)$ by Lemma 4.2 (5). This shows that $H \subseteq_2 K$, which contradicts Lemma 5.1.

Next, assume that $T[1] \subseteq O_2(H)$. Then $N^\infty(1; T) \subseteq T[1] \subseteq O_2(H)$ by Lemma 5.4 (2). Suppose that $|\mathscr{H}| = 1$. Since $T[2] \not\subseteq O_2(H) = C_T(V)$ by Lemma 5.11 (1) and Lemma 3.1 (1), $\overline{P} \cong Z_3$, Z_5, or Esp_{27} by Lemma 3.3. Thus $|S: T| \leq |S: O_2(H)| = |\overline{S}| \leq 2^4$ because $\overline{S}$ is isomorphic to a subgroup of $\mathrm{Aut}(\overline{P})$. Suppose that $|\mathscr{H}| \geq 2$ and that T is transitive on $\mathscr{H}$. Since $T = T[2]$ by Lemma 5.11 (2), T is generated by elements inducing transpositions on the set $\mathscr{H}$. So $|\mathscr{H}| = 2$, and $|W| = 2^2$ for all $W \in \mathscr{H}$. Hence $|S: T| \leq |S\ O_2(H)| = |\overline{S}| \leq 2^3$. We conclude that $|S: T| \leq 2^4$ if T is transitive on $\mathscr{H}$. Now, by Lemma 5.6, choose $X \in \Gamma_H(T)$ and $Y \in \Gamma_K(T)$ so that $Y \subseteq_2 X$ and $\langle X, Y \rangle$ is solvable. Then $N^\infty(1; T) \subseteq O_2(K)$ by Lemma 5.5 (2), which contradicts Lemma 5.4 (3). ∎

Let D be a maximal subgroup of S containing T so that D has precisely two orbits $\mathscr{H}_1$ and $\mathscr{H}_2$ on the set $\mathscr{H}$.

LEMMA 5.13. *There exists $Y \in \Gamma_K(D)$ such that $\langle Y, O^2(H) \rangle$ is solvable.*

Proof. Put $V_i = \langle \mathscr{H}_i \rangle$ for $i = 1, 2$. Then there exist D-invariant subgroups P_i $(i = 1, 2)$ of P containing $C_H(V)$ such that $\overline{P} = \overline{P_1 P_2}, [V_i, \overline{P_i}] = V_i, [V_i, \overline{P_{3-i}}] = 1$ $(i = 1, 2)$, and S interchanges $\overline{P_1}$ and $\overline{P_2}$ by Lemma 5.11 (1) and Lemma 3.3. Put $Z_i = DP_i$ $(i = 1, 2)$, and $U = C_{V_2}(D)$.

Let $Z \in \Gamma_K(D)$, and put $E = O_2(Z)$. Since $U \subseteq \Omega_1(Z(D)) \subseteq \Omega_1(Z(E))$ and $[O_2(Z), O^2(Z)] \subseteq E \subseteq D$, Lemma 4.4 shows that $[U, O^2(Z)] \subseteq [\Omega_1(Z(E)), O^2(Z)] = 1$ because $\Omega_1(Z(S)) \subseteq Z(K)$. Thus $[U, O^2(K)] = [U, \langle O^2(Z) | Z \in \Gamma_K(D) \rangle] = 1$ by Lemma 4.2 (6). Therefore $\langle Z_1, O^2(K) \rangle$ is solvable as it is contained in $C_G(U)$.

Since $H = \langle Z_1, S\rangle$ and Z_1 is generated by the set of its D-irreducible subgroups, there exists an element $Z' \in \Gamma_H(D)$ such that $Z' \subseteq Z_1$. Let $\Delta = \{Z'\} \cup \Gamma_K(D)$. Then, by Lemma 4.6, there exists an element $Y \in \Delta$ such that $Y \subseteq_2 X$ for all $X \in \Delta$. Suppose that $Y = Z'$. Then $[O_2(Z'), O^2(Z')] \subseteq \bigcap_{X \in \Gamma_K(D)} O_2(X) \subseteq O_2(K)$ by Lemma 4.2 (5), so $[O_2(H), O^2(H)] = [O_2(H), \langle O^2(Z')^S\rangle] = \langle [O_2(H), O^2(Z')]^S\rangle = \langle [O_2(Z'), O^2(Z')]^S\rangle \subseteq O_2(K)$ by Lemma 4.2 (6)(2). This shows that $H \subseteq_2 K$, a contradiction. Therefore $Y \in \Gamma_K(D)$ and $Y \subseteq_2 Z'$.

Let $E' = O_2(Z')$. Since $V_1 \subseteq \Omega_1(Z(E'))$ and $[O_2(Y), O^2(Y)] \subseteq E'$, Lemma 4.4 shows that $[V_1, O^2(Y)] \subseteq [\Omega_1(Z(E')), O^2(Y)] = 1$ because $\Omega_1(Z(S)) \subseteq Z(K)$. Thus $\langle O^2(H), D, O^2(Y)\rangle$ is solvable as it is contained in $N_G(V_1)$. ∎

Now, we reach a final contradiction.

Choose $Y \in \Gamma_K(D)$ so that $\langle Y, O^2(H)\rangle$ is solvable. Then there exist elements $X, Z \in \Gamma_H(D)$ such that $Z \not\subseteq_2 Y \not\subseteq_2 X$ by Lemma 4.3. Put $N = \langle X, Y, Z\rangle$. Since N is solvable and $|S: D| = 2$, Lemma 5.5 (2) shows that $N^{\infty}(1; D) \subseteq O_2(H) \cap O_2(K)$, which contradicts Lemma 5.4 (3). This completes the proof of Theorem 1.

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